Gauge Invariant Celestial Observables

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13th August 2022

Abstract

In this note, it is shown that the gravitational memory effects (in particular, the displacement and spin memory effects) can be computed as the holonomies associated to the (tree–level) expectation value of the gauge invariant gravitational Wilson loop, corresponding to a cosmic string intersecting past and future null infinities of an asymptotically flat spacetime.

1. Let $\langle \mathscr{M}, \mathbf{g} \rangle$ be a four–dimensional spacetime and $\mathcal{C} : \mathcal{I} \subseteq \mathbb{R} \longrightarrow \mathscr{M}$ a closed $\mathscr{C}^{>1}$–differentiable curve in $\mathscr{M}$ referred to on what follows simply as a *loop*. Furthermore, let $\Gamma_{\mu} \coloneqq (\Gamma_{\mu})^{\alpha}{}_{\beta}$ be the family of matrices whose entries constitute a representation of the Christoffel symbols associated to the Riemann–Levi-Civita connection $\nabla^{\mathbf{g}}$ determined by the Lorentzian metric tensor field $\mathbf{g} \coloneqq (g_{\mu\nu})$, and moreover, let us define the holonomy induced by the connection $\nabla^{\mathbf{g}}$ as a function of the *loop* $\mathcal{C}$ as follows[3, 4, 5, 6]:

$$\mathscr{H}ol^{\nabla^{\mathbf{g}}}(\mathcal{C}) \coloneqq \mathscr{P} \circ \exp\left(\oint_{\mathcal{C}} \Gamma_{\mu}(x)\, \mathbf{d}x^{\mu}\right), \tag{1}$$

where $\mathscr{P}$ is the path–ordering operator. Therefore, the *gravitational Wilson gauge–invariant operator* (called below simply as the *Wilson loop*) determined by the connection $\nabla^{\mathbf{g}}$ as a functional of the closed curve $\mathcal{C}$, describes the precession of the parallel translation of a vector along the *loop* $\mathcal{C}$, and is given explicitly as follows:

$$\mathscr{W}^{\nabla^{\mathbf{g}}}(\mathcal{C}) \coloneqq \mathrm{Tr}\left[\mathscr{H}ol^{\nabla^{\mathbf{g}}}(\mathcal{C})\right] - \dim(\mathscr{M}) = \mathrm{Tr}\left[\mathscr{P} \circ \exp\left(\oint_{\mathcal{C}} \Gamma_{\mu}(x)\, \mathbf{d}x^{\mu}\right)\right]. \tag{2}$$

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Our aim is to compute the perturbative expansion of the Wilson loop associated to a cosmic string intersecting the past and future null infinities $\mathscr{I}^{\pm}$. Let us then consider the expansion:

$$\mathcal{W}^{\nabla^{\mathbf{g}}}(\mathcal{C}) = \sum_{k=1}^{\infty} \mathcal{W}_{\mathcal{C}}^{(k)} = \mathcal{W}_{\mathcal{C}}^{(1)} + \mathcal{W}_{\mathcal{C}}^{(2)} + \mathcal{W}_{\mathcal{C}}^{(3)} + \mathcal{O}(4), \tag{3}$$

where:

$$\mathcal{W}_{\mathcal{C}}^{(1)} \coloneqq \oint_{\mathcal{C}} \Gamma_{\mu}(x)\, \mathbf{d}x^{\mu}, \tag{4}$$

$$\mathcal{W}_{\mathcal{C}}^{(2)} \coloneqq \frac{1}{2}\mathscr{P} \oint_{\mathcal{C}} \left(\oint_{\mathcal{C}} \left(\Gamma^{\alpha}_{\mu\beta}(x)\, \Gamma^{\beta}_{\nu\alpha}(x') \right) \mathbf{d}x'^{\nu} \right) \mathbf{d}x^{\mu}, \tag{5}$$

$$\mathcal{W}_{\mathcal{C}}^{(3)} \coloneqq \frac{1}{3!}\mathscr{P} \oint_{\mathcal{C}} \left(\oint_{\mathcal{C}} \left(\oint_{\mathcal{C}} \Gamma^{\alpha}_{\mu\beta}(x)\, \Gamma^{\beta}_{\nu\gamma}(x')\, \Gamma^{\gamma}_{\lambda\alpha}(x'')\, \mathbf{d}x''^{\lambda} \right) \mathbf{d}x'^{\nu} \right) \mathbf{d}x^{\mu}, \tag{6}$$

and thenceforth. Now, let us open a chart $\psi \coloneqq \langle \mathcal{U} \subseteq \mathscr{M}, x^{\mu} \rangle$ with spherical polar coordinates $x^{\mu} \coloneqq (t, r, \theta, \varphi)$ on the spacetime manifold $\mathscr{M}$, supporting a static and spherically symmetric solution to the Einstein equations, for which the metric tensor field $\mathbf{g}$ assumes the following form:

$$\mathbf{g} = A(r)\, \mathbf{d}t \otimes \mathbf{d}t + B(r)\, \mathbf{d}r \otimes \mathbf{d}r + C(r) \left(\mathbf{d}\theta \otimes \mathbf{d}\theta + \sin^2(\theta)\, \mathbf{d}\varphi \otimes \mathbf{d}\varphi \right). \tag{7}$$

Let $g_{\mu\nu} \coloneqq \mathbf{g}(\partial/\partial x^{\mu}, \partial/\partial x^{\nu})$ be the matrix whose entries are the components of the metric tensor field $\mathbf{g}$ with respect to the chart ψ. Therefore,

$$(g_{\mu\nu}) = \operatorname{diag}\left(A(r), B(r), C(r), C(r)\sin^2(\theta) \right), \tag{8}$$

whose inverse $(g^{\mu\nu}) \coloneqq (g^{-1})_{\mu\nu}$ is given by:

$$(g^{\mu\nu}) = \operatorname{diag}\left(\frac{1}{A(r)}, \frac{1}{B(r)}, \frac{1}{C(r)}, \frac{1}{C(r)\sin^2(\theta)} \right). \tag{9}$$

Let us recall that the components of the Christoffel symbols $\Gamma^{\alpha}_{\mu\nu}$ $(0 \leq \alpha, \mu, \nu \leq 3)$ associated to the Riemann–Levi-Civita connection $\nabla^{\mathbf{g}}$ with respect to the chart ψ are determined by the equation:

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\gamma} \left(\nabla^{\mathbf{g}}_{\nu} g_{\mu\gamma} + \nabla^{\mathbf{g}}_{\mu} g_{\nu\gamma} - \nabla^{\mathbf{g}}_{\gamma} g_{\mu\nu} \right). \tag{10}$$

Using Eqs. (8, 9, 10), a simple computation yields the following result:

$$\Gamma^{t}_{tr} = \Gamma^{t}_{rt} = \frac{A'(r)}{2A(r)}, \quad \Gamma^{r}_{tt} = -\frac{A'(r)}{2B(r)}, \quad \Gamma^{r}_{rr} = \frac{B'(r)}{2B(r)}, \tag{11}$$

$$\Gamma^{r}_{\theta\theta} = -\frac{C'(r)}{2B(r)}, \quad \Gamma^{r}_{\varphi\varphi} = -\frac{C'(r)\sin^2\theta}{2B(r)}, \quad \Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{C'(r)}{2C(r)}, \tag{12}$$

$$\Gamma^{\theta}_{\varphi\varphi} = -\frac{\sin(2\theta)}{2},\ \ \Gamma^{\varphi}_{\varphi r} = \Gamma^{\varphi}_{r\varphi} = \frac{C'(r)}{2C(r)},\ \ \Gamma^{\varphi}_{\varphi\theta} = \Gamma^{\varphi}_{\theta\varphi} = \cot(\theta)\,. \tag{13}$$

In order to model a *cosmic string intersecting the past and future null infinities* $\mathscr{I}^{\pm}$, let us introduce the following parametrisation for the *loop* $\mathcal{C}$ with respect to which the Wilson gauge–invariant operator $\mathcal{W}^{\nabla\mathbf{g}}(\mathcal{C})$ is computed:

$$\tau \in \mathcal{I} \subseteq \mathbb{R} \mapsto x^{\mu}(\tau) \coloneqq (t_0, r_0, \theta_0, \varphi)\,,\ \ \varphi : [0, 2\pi] \longrightarrow \mathbb{S}^1. \tag{14}$$

Let us introduce the following notation for the ordinary derivative of the $(\theta\theta)$–component $C(r) \coloneqq \mathbf{g}(\partial/\partial\theta, \partial/\partial\theta)$ of the metric tensor field $\mathbf{g}$,

$$C'(\tau_0) \coloneqq \left.\frac{dC(r)}{dr}\right|_{r=\tau_0}. \tag{15}$$

Therefore, with the assistance of Eqs. (11, 12, 13, 15), and employing the parametrisation given by Eq. (14), a simple computation using induction in $k \in \mathbb{N}$ gives the following general result for the perturbative expansion of the Wilson loop (cf. Eqs. (2, 3, 4, 5, 6)):

$$\mathcal{W}^{(2k+1)}_{\mathcal{C}} = 0,\ \ \mathcal{W}^{2(k+1)}_{\mathcal{C}} = \frac{(-1)^{k+1}\,2\,(2\pi)^{2(k+1)}}{(2k+2)!}\left(\cos^2(\theta_0) + \frac{\sin^2(\theta_0)\,(C'(r_0))^2}{4C(r_0)\,B(r_0)}\right)^{k+1}. \tag{16}$$

Lastly, summing the series determined by the terms obtained in Eq. (16) in the perturbative expansion of Eq. (3), one deduces the following generic form of the Wilson loop for a static and spherically symmetric solution $\langle \mathscr{M}, \mathbf{g}\rangle$ to the Einstein equations:

$$\mathcal{W}^{\nabla\mathbf{g}}(\mathcal{C}) = 2\left(\cos\left(\sqrt{\frac{4\pi^2\left(4B(r_0)\,C(r_0)\cos^2(\theta_0) + (C'(r_0))^2\sin^2(\theta_0)\right)}{4B(r_0)\,C(r_0)}}\right) - 1\right). \tag{17}$$

We are finally in the possession of the results which are necessary in order to compute the gauge invariant Wilson operator for the asymptotically flat line element:

$$\mathbf{d}\mathfrak{s}^2 \coloneqq g_{\mu\nu}\mathbf{d}x^{\mu}\mathbf{d}x^{\nu} = -A(r)\,(\mathbf{d}t)^2 + (\mathbf{d}r)^2 + A(r)\,(\mathbf{d}z)^2 + r^2B(r)\,(\mathbf{d}\varphi)^2\,, \tag{18}$$

corresponding to the gravitational field $\mathbf{g}$ supporting a cosmic string extending to the null infinites $\mathscr{I}^{\pm}$. Let us, then, suppose that the spacetime $\langle \mathscr{M}, \mathbf{g}\rangle$ is characterised by a cylindrical symmetry along the z–axis, the latter of which is the image of the integral curve of a Killing vector field $\partial/\partial z$. Therefore, the matrix given by Eq. (8) assumes the following form:

$$(g_{\mu\nu}) = \mathrm{diag}\left(-F(r), 1, F(r), r^2G(r)\right), \tag{19}$$

for some functions $F, G \in \mathscr{C}^{>1}(\mathbb{R})$. Replacing these components in Eqs. (15, 16, 17), one concludes that:

$$\mathscr{W}^{\nabla \mathbf{g}}(\mathcal{C}) = 2\left(\cosh\left(2\pi\sqrt{-G(r_0) - r_0 G'(r_0) - \frac{r_0^2 (G'(r_0))^2}{4G(r_0)}}\right) - 1\right). \tag{20}$$

2. Our objective in the present paper is to compute the gravitational memory effects associated to the gauge invariant Wilson loop operator $\mathscr{W}_{\mathcal{C}}^{\nabla \mathbf{g}}$ corresponding to a $\mathscr{C}^{>1}$–differentiable curve $\mathcal{C}$ intersecting $\mathscr{I}^{\pm}$ in an asymptotically flat spacetime $\langle \mathscr{M}, \mathbf{g} \rangle$. Before achieving this aim, we should briefly discuss the structure of the perturbative expansion of $\mathscr{W}_{\mathcal{C}}^{\nabla \mathbf{g}}$ in order to ascertain that the asymptotic value of the vacuum–to–vacuum expectation value of the quantum Wilson operator converges to the classical value obtained from the Einstein equations, as in §1. Let us write the metric tensor field $\mathbf{g}$ representing the gravitational field in the spacetime $\mathscr{M}$ in a *"freely–falling"* coordinate system x^{μ}, according to which $g_{\mu\nu}(x) \coloneqq \mathbf{g}\left(\partial/\partial x^{\mu}\big|_x, \partial/\partial x^{\nu}\big|_x\right)$ $(x \in \mathscr{M})$, whose inverse matrix is defined by $g^{\mu\nu} \coloneqq (g^{-1})_{\mu\nu}$, while the metric density is given by $\tilde{g}^{\mu\nu} \coloneqq g^{\mu\nu}\sqrt{-g}$, where $g \coloneqq \det(g_{\mu\nu})$. Let us denote by $h_{\mu\nu}$ the graviton field and $k\, h^{\mu\nu}$ a quantum perturbation with respect to the flat background metric $\eta \coloneqq \mathrm{diag}(-,+,+,+)$, with k is parametrising the energy scale of the quantum fluctuation, such that:

$$\tilde{g}^{\mu\nu}(x) = \eta^{\mu\nu} + k\, h^{\mu\nu}. \tag{21}$$

Then, one deduces the following expansion for the graviton field propagating on the flat background metric η,

$$\tilde{g}_{\mu\nu}(x) = \eta_{\mu\nu} - k\, h_{\mu\nu} + k^2\, h_{\mu\sigma} h^{\sigma}_{\nu} - k^3\, h_{\mu\sigma} h^{\sigma}_{\rho} h^{\rho}_{\nu} + \mathcal{O}\left(k^4\right). \tag{22}$$

Hence, the two–point Green function representing the propagator of the graviton field is given by the vacuum–to–vacuum expectation value:

$$\langle h_{\alpha\beta}(x)\, h_{\mu\nu}(x') \rangle. \tag{23}$$

Another important quantity in the description of the propagation of quantum excitations on the gravitational field is the two–point Green function associated to the Christoffel symbols $\Gamma^{\alpha}_{\mu\nu}(x)$ induced by the Riemann–Levi-Civita connection $\nabla^{\mathbf{g}} = \nabla^{\eta + k\mathbf{h}}$, for instance,

$$\left\langle \Gamma^{\alpha}_{\mu\beta}(x)\, \Gamma^{\beta}_{\nu\alpha}(x') \right\rangle. \tag{24}$$

Nonetheless, since our main concern regards the asymptotic behaviour of the gravitational

field, and the associated memory effects, in the past and future null infinities $\mathscr{I}^{\pm}$, it is not necessary to compute explicitly the expressions determined by Eqs. (23, 24). Let us thus verify that, indeed, the vacuum–to–vacuum expectation values of the Wilson loop converges (asymptotically) to the classical value given by General Relativity (GR) for the line element (recall Eq.(18)) of a cosmic string extending to $\mathscr{I}^{\pm}$. The action functional governing the evolution of an excitation of the quantum Wilson loop operator is the four–integral of the Einstein–Hilbert Lagrangian density:

$$S_{\mathrm{EH}} = \frac{2}{k^2}\int_{\mathscr{M}} \mathscr{R}\left(x\right)\, \mathbf{d}^4x, \tag{25}$$

where the quantum fluctuations are parametrised by the Planck mass, for which $k \coloneqq \sqrt{32\pi G_{\mathrm{N}}}$, and $\mathscr{R}\left(x\right)$ denotes the Ricci curvature scalar density[1] of the Riemann–Levi-Civita connection $\nabla^{\eta+k\mathbf{h}}$. Thence, the generating functional of the Einstein gravity can be written, formally, as follows:

$$\mathscr{Z}\left[T_{\mu\nu}\right] \propto \int \mathbf{M}\left[\tilde{g}^{\mu\nu}\right] \exp\left(i\int_{\mathscr{M}}\left(R - \frac{\left(\partial_\mu \tilde{g}^{\mu\nu}\right)^2}{\xi k^2} + \frac{\tilde{g}^{\mu\nu}T_{\mu\nu}}{k}\right)\mathbf{d}^4x\right)\left[\mathbf{d}\tilde{g}^{\mu\nu}\left(x\right)\right],$$

where $T_{\mu\nu}$ is the external source of the gravitational field $\tilde{g}^{\mu\nu}$, ξ is the gauge parameter, $\left(-\partial_\mu \tilde{g}^{\mu\nu}\right)^2/k^2$ is the gauge–fixing term and $\mathbf{M}\left[\tilde{g}^{\mu\nu}\right]$ represents the *formal* inverse of the graviton propagator. Now, let us return to Eqs. (2, 3, 4, 5, 6). Denoting the classical value of the Wilson loop determined by GR by the expression:

$$\mathscr{W}^{\nabla\mathbf{g}}\left(\mathcal{C}\right) = \delta^\beta_\alpha \oint_{\mathcal{C}} \Gamma^\alpha_{\mu\beta}\left(x\right)\, \mathbf{d}x^\mu + \frac{1}{2}\delta^\beta_\alpha \mathscr{P}\oint_{\mathcal{C}}\left(\oint_{\mathcal{C}}\left(\Gamma^\alpha_{\mu\beta}\left(x\right)\Gamma^\beta_{\nu\alpha}\left(x'\right)\right)\mathbf{d}x'^\nu\right)\mathbf{d}x^\mu + \mathcal{O}\left(k^6\right), \tag{26}$$

the perturbative expansion of the vacuum–to–vacuum expectation value of the gauge invariant gravitational Wilson operator can be written as follows:

$$\left\langle \mathscr{W}^{\nabla\mathbf{g}}\right\rangle_{\mathcal{C}} = \delta^\beta_\alpha \oint_{\mathcal{C}} \left\langle\Gamma^\alpha_{\mu\beta}\left(x\right)\right\rangle\, \mathbf{d}x^\mu + \frac{1}{2}\delta^\beta_\alpha \mathscr{P}\oint_{\mathcal{C}}\left(\oint_{\mathcal{C}}\left(\left\langle\Gamma^\alpha_{\mu\beta}\left(x\right)\Gamma^\beta_{\nu\alpha}\left(x'\right)\right\rangle\right)\mathbf{d}x'^\nu\right)\mathbf{d}x^\mu + \mathcal{O}\left(k^6\right). \tag{27}$$

Let us denote by $\overset{(n)}{\Gamma}{}^\alpha_{\mu\nu}$ $(n \geq 1)$ be the n^{th}–order term in the parameter k which is obtained by substituting the perturbation of the metric tensor given by Eq. (22) in the expression determining the Christoffel symbols, Eq. (10), so:

$$\Gamma^\alpha_{\mu\nu} = k\ \overset{(1)}{\Gamma}{}^\alpha_{\mu\nu} + k^2\ \overset{(2)}{\Gamma}{}^\alpha_{\mu\nu} + \ldots\ . \tag{28}$$

[1]The *Ricci curvature density* $\mathscr{R}\left(x\right)$ is related to the *Ricci curvature scalar* $R\left(x\right)$ by $\mathscr{R}\left(x\right) \coloneqq R\left(x\right)\sqrt{-g}$.

From this, a simple computation shows that the first order vacuum–to–vacuum expectation value of the Wilson loop is such that:

$$\left\langle \mathcal{W}_{\mathcal{C}}^{(1)} \right\rangle = \oint_{\mathcal{C}} \left(\frac{k}{2} \partial_\mu \left\langle h_\alpha^\alpha \left(x \right) \right\rangle - \frac{k^4}{4} \partial_\mu \left\langle h_{\alpha\beta} \left(x \right) h^{\alpha\beta} \left(x \right) \right\rangle + \mathcal{O}\left(k^6 \right) \right) \mathbf{d}^4 x = 0,$$

while the second order expectation value can be written as:

$$\left\langle \mathcal{W}_{\mathcal{C}}^{(2)} \right\rangle = \frac{1}{2} \mathcal{P} \oint \left(\oint \left(\left(k^2 \left\langle \overset{(1)}{\Gamma}{}_{\mu\lambda}^{\alpha} \left(x \right) \overset{(2)}{\Gamma}{}_{\nu\alpha}^{\lambda} \left(x' \right) \right\rangle + 2k^3 \left\langle \overset{(1)}{\Gamma}{}_{\mu\lambda}^{\alpha} \left(x \right) \overset{(2)}{\Gamma}{}_{\nu\alpha}^{\lambda} \left(x' \right) \right\rangle \right) \right) \mathbf{d}^4 x' \right) \mathbf{d}^4 x \tag{29}$$

$$+ \frac{1}{2} \mathcal{P} \oint \left(\oint \left(\left(k^4 \left\langle \overset{(2)}{\Gamma}{}_{\mu\lambda}^{\alpha} \left(x \right) \overset{(2)}{\Gamma}{}_{\nu\alpha}^{\lambda} \left(x' \right) \right\rangle + 2k^4 \left\langle \overset{(1)}{\Gamma}{}_{\mu\lambda}^{\alpha} \left(x \right) \overset{(3)}{\Gamma}{}_{\nu\alpha}^{\lambda} \left(x' \right) \right\rangle + \mathcal{O}\left(k^6 \right) \right) \right) \mathbf{d}^4 x' \right) \mathbf{d}^4 x \tag{30}$$

$$=: \left\langle \mathcal{W}^{(2)} \right\rangle_1 + \left\langle \mathcal{W}^{(2)} \right\rangle_2 + \left\langle \mathcal{W}^{(2)} \right\rangle_3 + \left\langle \mathcal{W}^{(2)} \right\rangle_4 + \mathcal{O}\left(k^6 \right). \tag{31}$$

Let us note that the term $\left\langle \mathcal{W}^{(2)} \right\rangle_1$ is the lowest order correction to $\left\langle \mathcal{W}_{\mathcal{C}}^{(2)} \right\rangle$, which is due to the graviton self–energy. Furthermore, on one hand, $\left\langle \mathcal{W}^{(2)} \right\rangle_2$ is a quadratic integral whose integrand is the vacuum–to–vacuum expectation value of a product between a $\overset{(1)}{\Gamma}$–term and $\overset{(2)}{\Gamma}$–term, while that, on the other hand, $\left\langle \mathcal{W}^{(2)} \right\rangle_3$ is a quadratic integral whose integrand is obtained by a product of two $\overset{(2)}{\Gamma}$–terms. Therefore, the lowest order contribution of them to $\left\langle \mathcal{W}^{\nabla^{\eta+k}\mathbf{h}} \right\rangle$ is of order k^4. We thus conclude that the asymptotic value of the vacuum–to–vacuum expectation value of the Wilson loop approaches the value obtained by GR. Thus, we can proceed to compute the memory effects associated to the excitations of the gravitational field propagating to $\mathscr{I}^{\pm}$ using the results obtained in the last Section to the metric field associated to a cosmic string piercing the past and future null infinities.

3. Based on the results of **§2,** let us model the perturbation of the gravitational field as a classical gravitational wave (GW), and moreover, in order to facilitate the computation of the holonomies associated to the parallel translation of tangent vectors in the spacetime supporting the propagation of a GW in an asymptotically flat spacetime background, let us introduce an orthonormal frame field $\left(\mathbf{g}^{\hat{\mu}} : 0 \leq \mu \leq 3 \right)$ for which:

$$\mathbf{g} = -\mathbf{g}^0 \otimes \mathbf{g}^0 + \sum_{i=1}^{3} \mathbf{g}^i \otimes \mathbf{g}^i. \tag{32}$$

For the purposes of our calculation, we adopt the *source oriented frame*[2, 8, 9], which is given

(asymptotically) by:

$$\mathbf{g}^{\hat{0}} = \left(1 - \frac{M}{r}\right)\mathbf{d}u + \left(1 + \frac{M_0}{r}\right)\mathbf{d}r - \frac{1}{r}\mathbf{d}\theta^A \int_{u_0}^{u} \partial_A M \, \mathbf{d}u', \tag{33}$$

$$\mathbf{g}^{\hat{r}} = -\frac{M - M_0}{r}\mathbf{d}u + \left(1 + \frac{M_0}{r}\right)\mathbf{d}r - \frac{1}{r}\mathbf{d}\theta^A \int_{u_0}^{u} \partial_A m \, \mathbf{d}u', \tag{34}$$

$$\mathbf{g}^{\hat{A}} = E_B{}^{\hat{A}} \left(\frac{1}{r} D_C C^{BC} \mathbf{d}u + \left(r\delta_A^B + \frac{1}{2} C_A{}^B\right)\mathbf{d}\theta^A\right). \tag{35}$$

Here, we denote by $E_A{}^{\hat{A}} = E_A{}^{\hat{A}}\left(\theta^B\right)$ a time–dependent dyad on the sphere for which:

$$\gamma^{AB} E^{\hat{A}}_{\ A} E^{\hat{B}}_{\ B} = \delta^{\hat{A}\hat{B}}. \tag{36}$$

Furthermore, the spin connection coefficients $\omega \coloneqq \overline{\omega} + \tilde{\omega}$ are determined as follows, from the Cartan first structural equation:

$$\overline{\omega}_{\hat{r}\hat{A}} = -E_{\hat{A}A}\mathbf{d}\theta^A, \ \ \overline{\omega}_{\hat{A}\hat{B}} = E_{\hat{A}}{}^B D_B E_{\hat{B}C}\mathbf{d}\theta^C, \tag{37}$$

$$\tilde{\omega}_{\hat{r}\hat{A}} = E_{\hat{A}}{}^A \left(\frac{1}{4r^2} N_{AB} D_C C^{BC}\mathbf{d}u + \frac{1}{2r^2} D^B C_{AB}\mathbf{d}r + \frac{1}{2} N_{AB}\mathbf{d}\theta^B\right), \tag{38}$$

$$\tilde{\omega}_{\hat{A}\hat{B}} = \left(\frac{1}{2r} D^A \tilde{C}_{AB}\mathbf{d}\theta^B + \frac{1}{4r^2}\left(\frac{1}{2} N_{AB}\tilde{C}^{AB} - D_A D_B \tilde{C}^{AB}\right)\mathbf{d}u\right)\varepsilon_{\hat{A}\hat{B}} + \mathcal{O}\left(\frac{1}{r^3}\right)\mathbf{d}r. \tag{39}$$

In order to compute the *displacement memory effect*[1, 9], we should consider the holonomy obtained by the parallel translation of a vector along trajectories of two test masses with initial relative physical distance $d_0 \coloneqq x\left(u_0\right)$ and velocity $v_0 \coloneqq v\left(u_0\right)$. Moreover, we suppose that a GW is emitted during the interval of proper–time $u_0 < u < u_1$. Thence, the parallel transportation of a vector along the geodesics of these two bodies is given by the holonomy:

$$\left(\mathcal{H}o\ell\right)^{\hat{A}} = \int_0^{x(u)} E^{\hat{A}}_{\ A}\left(u_1\right)\mathbf{d}\theta^A - \int_0^{d_0} E^{\hat{A}}_{\ A}\left(u_0\right)\mathbf{d}\theta^A \tag{40}$$

$$+ \int_{u_0}^{u} E^{\hat{A}}_{\ u}\left(d_0\right)\mathbf{d}u + \left(\int_{d_0}^{x(u)} E^{\hat{A}}_{\ A}\mathbf{d}\theta^A - \int_{u_0}^{u} E^{\hat{A}}_{\ u}\left(x\right)\mathbf{d}u\right). \tag{41}$$

A direct computation yields, to leading order,

$$\Delta\left(\mathcal{H}o\ell\right)^{\hat{A}} \simeq \frac{1}{2}\left(X_0{}^{\hat{B}}\Delta C^{\hat{A}}_{\ \hat{B}} - \beta^{\hat{B}}\int_{u_0}^{u_1}\left(C^{\hat{A}}_{\ \hat{B}}\left(u\right) - C^{\hat{A}}_{\ \hat{B}}\left(u_1\right)\right)\mathbf{d}u\right), \tag{42}$$

which is exactly the leading order of the displacement memory obtained by Strominger and

Zhiboedov. Finally, in order to compute the gyroscopic and spin memory effects, one must consider the effect of gravitational radiation on the rotational holonomy, which can be written, to leading order, as follows:

$$(\mathcal{H}o\ell)^{\hat{A}\hat{B}} = -\frac{1}{2r^2}\left(\int\left(D_A D_B \tilde{C}^{AB} - \frac{1}{2}N_{AB}\tilde{C}^{AB}\right)\mathbf{d}u - 2\int rD^B\tilde{C}_{AB}\mathbf{d}r\right). \tag{43}$$

Now, let us contemplate the following two possibilities.

(a) *Gyroscopic memory effect.* Supposing that the evolution of the freely falling bodies is such that the initial velocity is zero at some proper–time $u_0 \leq u \leq u_1$, and noting that the path upon which the integration of Eq. (43) is taken carried out is a timelike geodesic, we conclude that the angular momentum remains at order $\mathcal{O}(r^{-2})$, and thence, the leading order term becomes:

$$\Delta(\mathcal{H}o\ell)^{\hat{A}\hat{B}} = -\frac{1}{2r^2}\left(\int\left(D_A D_B \tilde{C}^{AB} - \frac{1}{2}N_{AB}\tilde{C}^{AB}\right)\mathbf{d}u\right)\varepsilon^{\hat{A}\hat{B}}, \tag{44}$$

a result that should be recognised as the gyroscopic memory effect.

(b) *Spin memory effect[7].* Let us assume that the proper–length of the path over which the integral of Eq. (43) is defined, let us say $\ell > 0$, is much smaller than the characteristic wavelength λ of the GW, that is to say, $\ell/\lambda << 1$. Under these conditions, one deduces the second term in Eq. (44) is, with respect to the factor ℓ/λ, smaller than the first by an order of magnitude, so that the leading order term of the rotational holonomy becomes[2]:

$$\Delta(\mathcal{H}o\ell)^{\hat{A}\hat{B}} = -\frac{1}{2r^2}\left(\int\left(D_A D_B \tilde{C}^{AB}\right)\mathbf{d}u\right)\varepsilon^{\hat{A}\hat{B}} \simeq -\frac{\Delta T}{2r^2}\left(D_A D_B \tilde{C}^{AB}\right)\varepsilon^{\hat{A}\hat{B}}, \tag{45}$$

which we identify as being proportional to the spin memory effect.

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